STUDIES IN CLIMATE DYNAMICS FOR ENVIRONMENTAL SECURITY: A CALIBRATED ANALYTICAL MODEL FOR THE THERMOHALINE AND WIND-DRIVEN CIRCULATION IN THE INTERIOR OF A SUBTROPICAL OCEAN

R. C. Alexander

A Report prepared for

ADVANCED RESEARCH PROJECTS AGENCY

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PREFACE

Meteorological studies suggest that technologically feasible operations might trigger substantial changes in the climate over broad regions of the globe. Depending on their character, location, and scale, these changes might be both deleterious and irreversible. If a foreign power were to bring about such perturbations either overtly or covertly, either maliciously or heedlessly, the results might be seriously detrimental to the security and welfare of this country. So that the United States may react rationally and effectively to any such actions, it is essential that we have the capability to: (1) evaluate all consequences of a variety of possible actions that might modify the climate, (2) detect trends in the global circulation that presage changes in the climate, either natural or artificial, and (3) determine, if possible, means to counter potentially deleterious climatic changes. Our possession of this capability would make incautious experimentation unnecessary, and would tend to deter malicious manipulation. To this end, the Advanced Research Projects Agency initiated a study of the dynamics of climate to evaluate the effect on climate of environmental perturbations. The present Memorandum is a technical contribution to this larger study.

Middle latitude (subtropical) ocean regions of the world form a vast and important connecting link to the atmospheric heat engine, and hence to the dynamics of climate, via exchanges of heat, momentum, and water vapor at the air/sea interface. This Memorandum reports on a study of a semiempirical model of the circulation immediately beneath the surface frictional layer in the open ocean. The model provides some insight into the physical and dynamical nature of the subsurface circulation, and thereby may prove useful in the more detailed numerical simulation of the oceanic system.

Other Rand publications that provide background information on the physics and dynamics of oceanic circulations include RM-5594-NSF, RM-6110-RC, RM-6210-ARPA, and RM-6211-ARPA.

ABSTRACT

An exponential solution to the thermocline equations is scaled and calibrated for the intermediate circulation in middle latitudes of the open ocean in accordance with the scaling and results of the more complete numerical models of Bryan and Cox. Evaluation of the average potential vorticity for a number of stations is a main feature of the calibration method. The results of tests on the calibrated model in which the wind stress parameter E is varied show good qualitative agreement with the published numerical results for the region in which the analytical model is valid.

The effects of varying the vertical thermal diffusion parameter K are investigated. A significant feature of the results is the lack of appreciable difference in temperature distributions for the cases when $E = K = 1$ (the reference case) and $E = 1$, $K = 0$ (the purely advective case). The relative sizes of terms in the heat-balance equation are investigated. It is shown that the dominant balance near the surface (but beneath the frictional layer) is between horizontal and vertical advection if the ratio $E/KC \gg 1$, where C is the dimensionless thermocline scale parameter evaluated from the potential vorticity equation. The value of the above ratio is approximately 5 in the reference case. For all values of the parameters it is shown that the ratio of horizontal advection to vertical diffusion is not less than one just beneath the surface layer in middle latitude regions of surface downwelling.

CONTENTS

I. INTRODUCTION

Analytical solutions to various forms of the steady thermocline equations have been found by Welander [1959], Fofonoff [1962], Blandford [1965], and Needler [1967]. The equations and solutions are restricted to certain regions of the open ocean beneath the surface frictional layer.

It is not clear what constitutes a proper set of boundary conditions for these solutions. As Veronis [1969] pointed out in a review article, the solutions contain certain arbitrary functions of integration which can be over-determined by specifying too many equally reasonable conditions. Moreover, the necessary role played by vertical diffusion of (apparent) heat in the solutions is questioned. It is clear from Veronis's review that what is needed is a more definite connection between the thermocline solutions and real ocean circulations.

With the publication of numerical solutions to a more complete set of equations [Bryan and Cox, 1967, 1968] it becomes possible to treat the analytical solutions semiempirically. Results of the numerical experiments suggest an appropriate analytical model and boundary condition, and provide a means of calibrating the remaining uncertainties and then testing the calibrated model.

The model equations employed in the present study are the thermocline equations derived by Robinson [1960]. Their validity for the open ocean in middle latitudes beneath the surface frictional layer is verified by Bryan and Cox [1968], in which the magnitudes of various terms in the heat- and vorticity-balance equations are considered.

The solution chosen here to the model equations is perhaps the simplest known solution capable of giving realistic results in qualitative agreement with those of the above numerical studies: an exponential-type solution for the temperature corresponding to the purely baroclinic flow in an infinitely deep ocean.

The boundary condition to be applied is the Ekman condition on vertical velocity at the base of the frictional layer as prescribed by a given wind-stress distribution at the surface. It is clear from Bryan and Cox [1967] that the value of a certain wind parameter must

play an important role in the circulation. The remaining degrees of freedom are calibrated with respect to Bryan and Cox's numerical studies.

One difficulty in Bryan and Cox's numerical model can be partially removed in the analytical model. Because of the complexity of the more complete model, Bryan and Cox found it necessary to set the dimensionless vertical-diffusion coefficient equal to one (except for regions of convective instability in higher latitudes, where it was effectively infinite). In the following, the diffusion coefficient will be allowed to take on various constant values.

The study consists of three phases: (1) scaling and calibration; (2) variation of the wind parameter to check the calibrated model; and (3) variation of the thermal-diffusion parameter with discussion and interpretation of new results in terms of heat-balance components.

II. THE MODEL

SCALED EQUATIONS

Let the x-axis be directed eastward along the equator, the y-axis northward, and the z-axis vertically upwards from a level surface (assumed plane) just beneath the surface frictional layer. Robinson [1960] derived the approximate governing equations appropriate to extraequatorial regions, based on a scale analysis of dimensionless numbers relevant to large-scale oceanic circulations. The Boussinesq approximation is employed in Robinson's derivation, and horizontal eddy transport processes are neglected. The resulting equations, written in dimensionless form, are

$$fv = \frac{\partial p}{\partial x} \quad , \tag{2.1}$$

$$fu = -\frac{\partial p}{\partial y} \quad , \tag{2.2}$$

$$-\rho = \frac{\partial p}{\partial z} \quad , \tag{2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad , \tag{2.4}$$

$$\rho = -\theta \quad , \tag{2.5}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = K\frac{\partial^2 \theta}{\partial z^2} \quad , \tag{2.6}$$

where Eq. (2.5) relates the perturbation density ρ to a dimensionless function $\theta(x,y,z)$ that will be called the temperature.

The above symbols, except for K, are defined in Table 1. The numerical values shown are appropriate to the North Atlantic or North Pacific. The scale constants (except for D, see below) have been chosen in accordance with the scaling given by Bryan and Cox [1967].

Table 1

DEFINITION OF SYMBOLS

(See text for definitions of the parameters K, E, and C.)

Dimensionless Quantity[*]	Physical Meaning	Scale Constant	Representative Scale Value
x, y	Horizontal coordinates y = ϕ, latitude	a, earth's radius	6.37×10^8 cm
z	Vertical coordinate	D, depth scale	300 m
u, v	x, y velocities	V_t (see below)	1.62 cm/sec
w	z velocity	$V_t D/a$	0.81×10^{-4} cm/sec
p	Perturbation pressure (departure from hydrostatic)	$\rho\alpha\Delta\theta gD$	1.45×10^5 dynes/cm^2
ρ	Perturbation density	$\rho_o\alpha\Delta\theta$	4.6×10^{-3} gm/cm^3
θ	Apparent temperature[**]	$\Delta\theta$	18^oC
τ^x	Zonal wind stress	$\omega a/2\pi$	1 dyne/cm^2
f = sin ϕ	Coriolis parameter	2Ω	1.46×10^{-4}/sec
β = cos ϕ	df/dϕ	$2\Omega/a$	2.29×10^{-13}/cm-sec

Other constants	Physical Meaning	Representative Value
V_t	$\alpha\Delta\theta gD/2\Omega a$, thermal-scale velocity	1.62 cm/sec
V_w	$\omega a/2\Omega\rho_o D$, wind-scale velocity	1.25 cm/sec
ω	Maximum of $\partial\tau^x/\partial y$	10^{-8} dyne/cm^3
d	$(2\Omega\kappa a^2/\alpha\Delta\theta g)^{1/3}$, vertical thermal diffusion scale	320 m[†]
κ	Vertical eddy thermal diffusivity	2.5 cm^2/sec[†]
α	Coefficient of thermal expansion	2.5×10^{-4}/deg C
ρ_o	Density of deep water at atmospheric pressure	1.03 gm/cm^3
g	Gravity	980 cm/sec^2

[*]Dimensional functions and variables are obtained by multiplying the dimensionless quantities by the appropriate scale constants.

[**]A linear combination of temperature and salinity. See Fofonoff (1962), p. 368.

[†]The value of κ is highly uncertain. See text.

PARAMETERS

The dimensionless thermal diffusion coefficient K in Eq. (2.6) is the first of three parameters used in the present analysis:

$$K = \frac{\kappa a}{V_t D^2} \quad \text{(thermal diffusion parameter)} \quad (2.7)$$

$$E = \frac{V_w}{V_t} \quad \text{(wind parameter)} \quad (2.8)$$

$$C = \frac{C^* D}{2\Omega} \quad \text{(exponential scale parameter} \quad (2.9)$$
$$\sim \text{thermocline scale)}$$

where κ is an assumed constant coefficient of vertical eddy thermal dif-fusivity. Its value is uncertain and varies with geographic location. Veronis [1969] gives estimates of $0.2 \lesssim \kappa \lesssim 2.0$ cm^2/sec for the thermo-cline region based on the models and analyses of others. Veronis also points out that $\kappa > 0$ is merely a sufficient condition to give deep upwelling.

As will be seen below, the wind parameter E results from applica-tion of the Ekman condition on surface vertical velocity. As seen from the definitions of V_w and V_t in Table 1, $E \sim 1/D^2$. The depth scale D has been deliberately left undefined; however, when $D = d$ (the thermal diffusion scale, see Table 1), then $K = 1$ and $E = \gamma - 1$, where γ is Bryan and Cox's [1967, 1968] wind parameter. These authors found that $\gamma \simeq 2$ can give results that realistically model the ocean circulation. Thus, in the present analysis there is the possibility of a realistic model when $K \simeq E \simeq 1$.

The exponential scale parameter C arises from the form of solution assumed below to the system of equations (2.1) through (2.6). Both C and its dimensional counterpart C^* are arbitrary. The exponential or thermocline scale depth is $2\Omega/C^*$. This constant will be employed to calibrate the model when $K = E = 1$.

MODEL BASIN AND SURFACE BOUNDARY CONDITION

The model basin and applied wind stress chosen are similar to that of Bryan and Cox [1967]. See Fig. 1. The dimensions of the basin are

and

$$0 \leq x \leq 1 \ ,$$

$$10° = \pi/18 \leq y = \phi \leq 1 + \pi/18 \simeq 67.3° \quad . \qquad \Biggr\} \qquad (2.10)$$

The depth of the model ocean is taken to be infinite.

The north/south component of wind stress is neglected and the east/west component is taken to be

where

$$\tau^x = - \frac{1}{2\pi} \sin 2\pi \ (\phi - \phi_o) \quad ,$$

$$\phi_o = \frac{\pi}{18} - \frac{1}{8} \quad . \qquad \Biggr\} \qquad (2.11)$$

The Ekman boundary condition is

$$w(x,y,0) = w_E \quad , \qquad (2.12)$$

where

and

$$w_E = E \ \hat{k} \ \cdot \ \text{curl} \left(\frac{\underset{\sim}{\tau}}{f} \right) = E\frac{\beta}{f^2} \ F(\phi) \quad ,$$

$$F(\phi) = \tau^x(\phi) - \tan \phi \ \frac{d\tau^x}{d\phi} \quad . \qquad \Biggr\} \qquad (2.13)$$

F is the Sverdrup stress function.

EXACT SOLUTION

There exists a fairly general class of exact exponential-type solutions to the system of equations (2.1) through (2.6). See Needler [1967] and the review article by Veronis [1969]. The simplest of these, correspond-
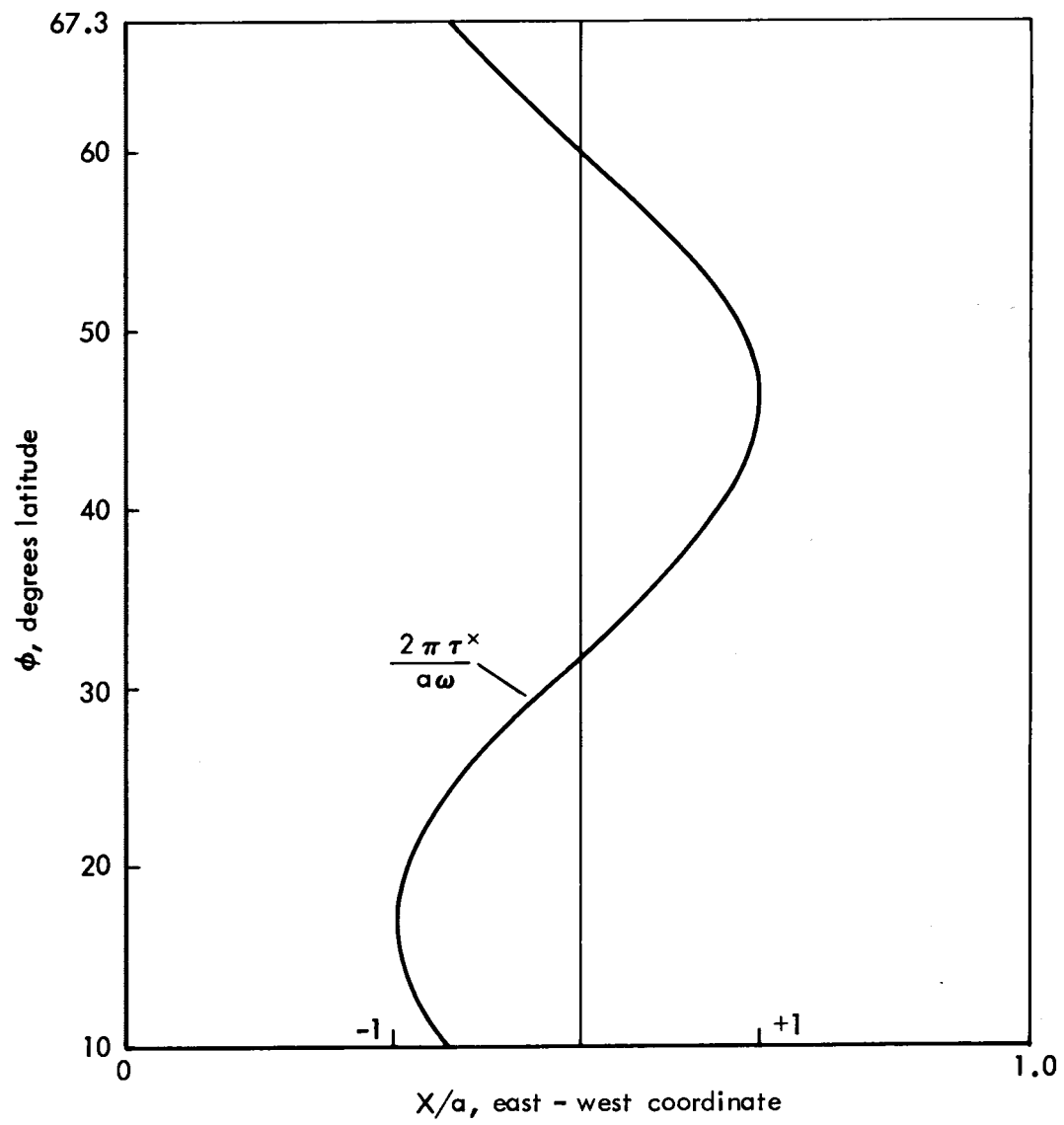
Fig. 1 -- Model basin and x-component τ^x of applied wind stress.

ing to the purely baroclinic flow in an infinitely deep ocean, is chosen for the present model. The temperature and velocities are

$$\theta = \theta_o(x,y) \ e^{Cz/f} \ , \tag{2.14}$$

$$u = - \frac{1}{C} \left[\frac{\partial \theta_o}{\partial y} + \frac{\partial \theta_o}{f} (1 - \frac{Cz}{f}) \right] \ e^{Cz/f} \ , \tag{2.15}$$

$$v = \frac{1}{C} \frac{\partial \theta_o}{\partial x} \ e^{Cz/f} \ , \tag{2.16}$$

$$w = \frac{CK}{f} + \frac{\beta}{C^2} \frac{\partial \theta_o}{\partial x} \ e^{Cz/f} \ , \tag{2.17}$$

in which the function $\theta_o \geq 0$ and the constant $C > 0$ are arbitrary. The quantity CK/f in Eq. (2.17) is the deep asymptotic vertical velocity w_∞ [Blandford, 1965].

SURFACE TEMPERATURE

The function θ_o is the temperature immediately beneath the surface frictional layer. This will be called the surface temperature.

Application of the boundary condition Eq. (2.12) to the vertical velocity Eq. (2.17) gives an equation for $\partial \theta_o / \partial x$ which can be integrated. One finds

$$\theta_o = \theta_e(\phi) + \frac{C^2}{f} \left[\frac{CK}{\beta} - \frac{EF(\phi)}{f} \right] (1-x) \ , \tag{2.18}$$

where an arbitrary function of latitude has been associated with the eastern-boundary surface temperature θ_e (more precisely, the surface temperature just outside the eastern boundary region). The quantity in brackets is proportional to $w_\infty - w_E$. As shown by Needler [1967], $\partial \theta / \partial x$ is proportional to this difference.

III. CALIBRATION

EASTERN BOUNDARY SURFACE TEMPERATURE

We set

$$\theta_e = A \cos^2 \phi \, , \tag{3.1}$$

where A is a normalizing constant chosen to make the maximum value of θ_o equal to one. Thus, the form of θ_e is fixed, but its amplitude will depend on the values of the parameters.

The $\cos^2 \phi$ form for surface temperature is reasonable and was used by Bryan and Cox [1967]. However, θ_e is strictly the temperature beneath the surface frictional layer just outside the eastern boundary region. Equation (3.1) should not be confused with a surface-temperature boundary condition.

It is expected that the results will not be too sensitive to any reasonable form assumed for θ_e. Compare Bryan and Cox [1967] with Bryan and Cox [1968] in which a linear distribution (in ϕ) of surface temperature was assumed. In the present analysis several test cases were computed in which a linear function was employed in place of Eq. (3.1). Only minor quantitative differences were found.

One difficulty in the present model is that θ_o should be "normalized" [using the constant A in Eq. (3.1)] to some value less than one when making detailed comparisons with the numerical models of Bryan and Cox [1967, 1968]. In the latter, the maximum θ occurs at the model sea surface but not below the surface where comparisons with the present formulation can be made. Sample calculations indicate that choices of $\theta_{max} \simeq 0.6$ to 0.7 give more accurate quantitative agreement with Bryan and Cox beneath the surface layer (z = -0.5d to -0.6d) in middle latitudes. However, this approach was abandoned as being too subjective, and it was decided to make $\theta_{max} = 1$. Qualitative comparisons with Bryan and Cox can still be made even though values of temperature may differ by 30 to 40 percent. (In any event, the real ocean has a well-mixed surface layer in which temperature is nearly uniform in the vertical. Therefore a maximum temperature of "one" occurring at the sea

surface would also occur at the base of the mixed layer where the present model is thought to apply.)

THERMOCLINE SCALE CONSTANT, C

Several methods were employed to evaluate C. The best of these was found by noting that

$$f \frac{\partial \theta}{\partial z} = C\theta \quad . \tag{3.2}$$

The left-hand side of the above represents a form of potential vorticity that is conserved (following a fluid parcel) in the model with K = 0, and may be approximately conserved for certain regions of the real ocean [Phillips, 1963]. If both sides of Eq. (3.2) are divided by θ the resulting expression can be employed to evaluate C.

Bryan and Cox [1968] computed the left-hand side of Eq. (3.2) from their numerical model and tabulated the results for several locations. If their values are divided by the arithmetic average temperature for each temperature interval, the resulting numbers are as shown in Table 2.

The numbers used to calculate C are denoted by asterisks. The values at 10°N are too close to the equator, and the upper values at 20° and 30°N are too close to the frictional layer. From the six values used, calculations of the average and standard deviation give C = 0.19 ± 0.04. Since only one-figure accuracy is considered justified, the value

$$C = 0.2 \tag{3.3}$$

was chosen, and is held fixed throughout the remainder of this study.

A second method employed to calibrate C in accordance with the Bryan and Cox studies consisted of measurement of exponential-scale depths. A third method involved setting E = K = 1 and computing surface temperature distributions for various C. A dominant feature of the results is found to be an east/west temperature gradient in low latitudes. This was compared to the corresponding temperature gradient occurring

Table 2

VALUES OF sin ϕ $\Delta\theta/\bar{\theta}\Delta z$ COMPUTED FROM TABLE 4

OF BRYAN AND COX (1968, part I).

(The values denoted by an asterisk
were used to calculate C.)

Isotherm Interval	Latitude			
	10°	20°	30°	40°
0.6 to 0.5	0.16			
0.5 to 0.4	0.10	0.16		
0.4 to 0.3	0.12	0.16*	0.17	
0.3 to 0.2	0.15	0.14*	0.15*	
0.2 to 0.1	0.09	0.22*	0.23*	0.24*

beneath the surface in the numerical models, but some subjective judgment was required to decide at what depth to make comparisons. The total range of values found from all methods was $0.14 \leq C \leq 0.29$.

The above method employing Eq. (3.2) can be used to calibrate C with respect to oceanographic data. Similar calculations were made for three hydrographic stations in the central North Atlantic. The average and standard deviation of six values gave $C = 0.19 \pm 0.03$ if a scale depth $D = 340$ m is used (0.17 ± 0.02 for $D = 300$ m).

The dimensionless exponential decay depth is $(\sin \phi)/C$. For $D = 340$ m and $C = 0.2$, this gives a depth of 850 m at 30°N. If the temperature at that latitude is 20°C at 150 m depth, the temperature at 1000 m depth is about $20/e \simeq 7$°C.

It is worth noting that the numbers in Table 2 exhibit a significantly smaller percentage of scatter than do the numbers on which they are based (departures of ±25 percent from the average of all numbers as compared to ±40 percent). Dividing by the average temperature for each interval has the effect of smoothing the results.

IV. RESULTS

VARIATION OF THE WIND PARAMETER, E

The purpose here is to test the calibrated model against published results from the numerical models. We hold $K = 1$ and $C = 0.2$ fixed, set $E = 0, 1, 3$ ($\gamma = 1, 2, 4$), and compare temperature distributions with Bryan and Cox [1967]. The results are shown in Figs. 2 through 4.

Good qualitative agreement is found in middle and lower latitudes, below about 50°N. (Latitudes below about 10°N are excluded from both models.) Agreement is fair to poor in higher latitudes. This is to be expected because of the convective instability in higher latitudes as found in the numerical model but not accounted for in the analytical model. In Fig. 2c (strong-wind case) the analytical model prescribes negative temperatures at the surface (density greater than that of deep water) to the northwest of the zero-degree isotherm.

For increasing E, some points of agreement are:

1. There is an increasing east/west temperature gradient in lower latitudes corresponding to a more pronounced subtropical gyre.

2. There is an increasing north/south temperature gradient in middle latitudes near the western boundary.

3. Colder water appears at the surface to the northwest.

Some points of disagreement are:

1. There are discrepancies in the temperature distribution in higher latitudes as noted above. Such discrepancies may be related to the temperature condition imposed in the numerical model in which sur-face temperature is a prescribed function of latitude. Improved quali-tative agreement is found between the reference case $E = K = 1$ and the numerical results of Bryan [1969] in which the surface temperature con-dition is relaxed. However, in the latter study a more detailed equa-tion of state is used; therefore, comparisons of the apparent temperature of the analytical model with the thermodynamic temperature of the more recent numerical model are not strictly valid.

2. There are discrepancies in the distribution in the immediate vicinity of lateral boundaries. Such discrepancies are to be expected because boundary processes are not accounted for in the analytical model.
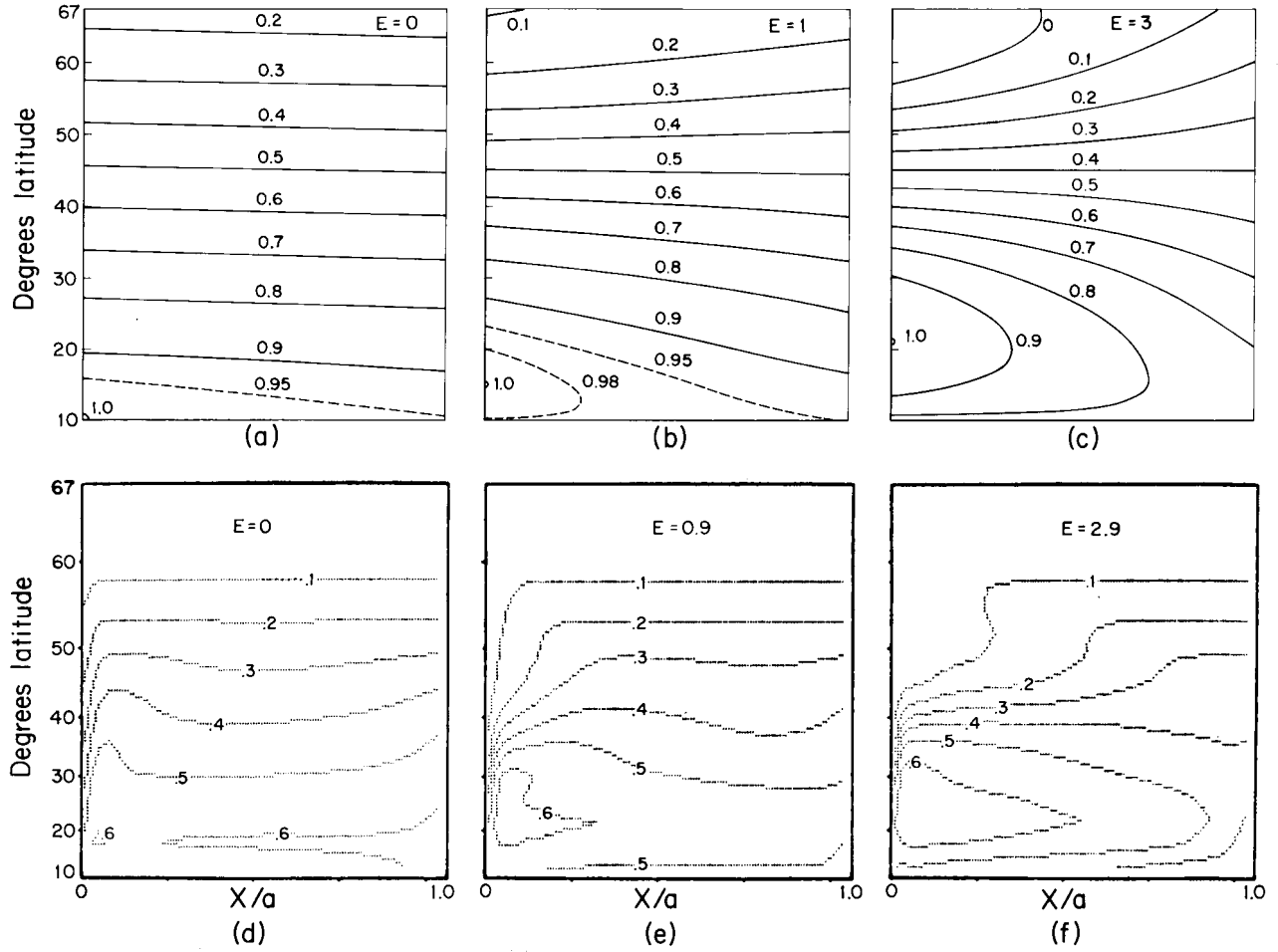
Fig. 2 -- Horizontal temperature patterns at z = 0 from the analytical model (a through c) compared with the numerical results of Bryan and Cox [1967] at z = -0.6d (d through f). From left to right E = 0 corresponds to no wind, E = 1 to moderate wind, E = 3 to strong wind. The parameters K = 1 and C = 0.2 are held fixed.
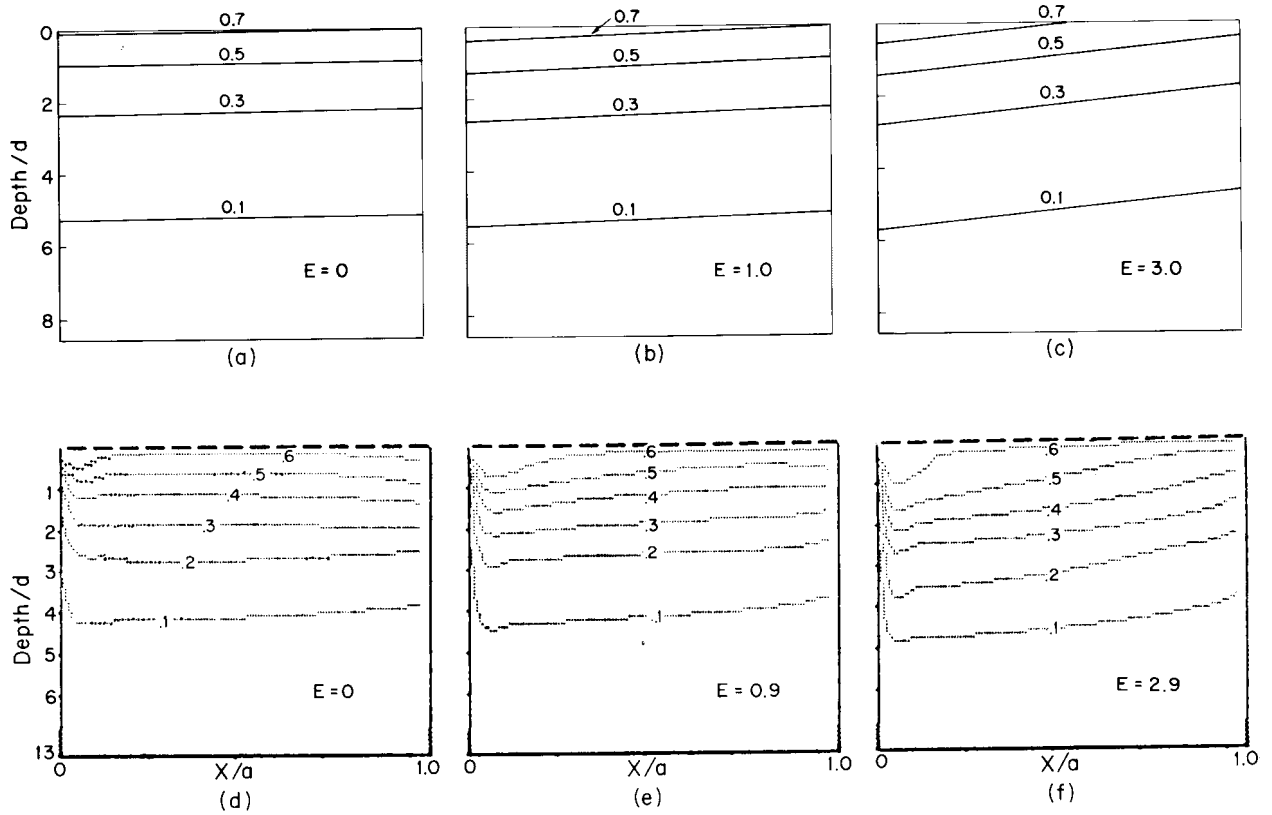
Fig. 3 -- East/west temperature sections at 32°N from the analytical model (a through c) compared with the numerical results (d through f) of Bryan and Cox [1967]. From left to right E = 0 corresponds to no wind, E = 1 to moderate wind, and E = 3 to strong wind. The parameters K = 1 and C = 0.2 are held fixed.
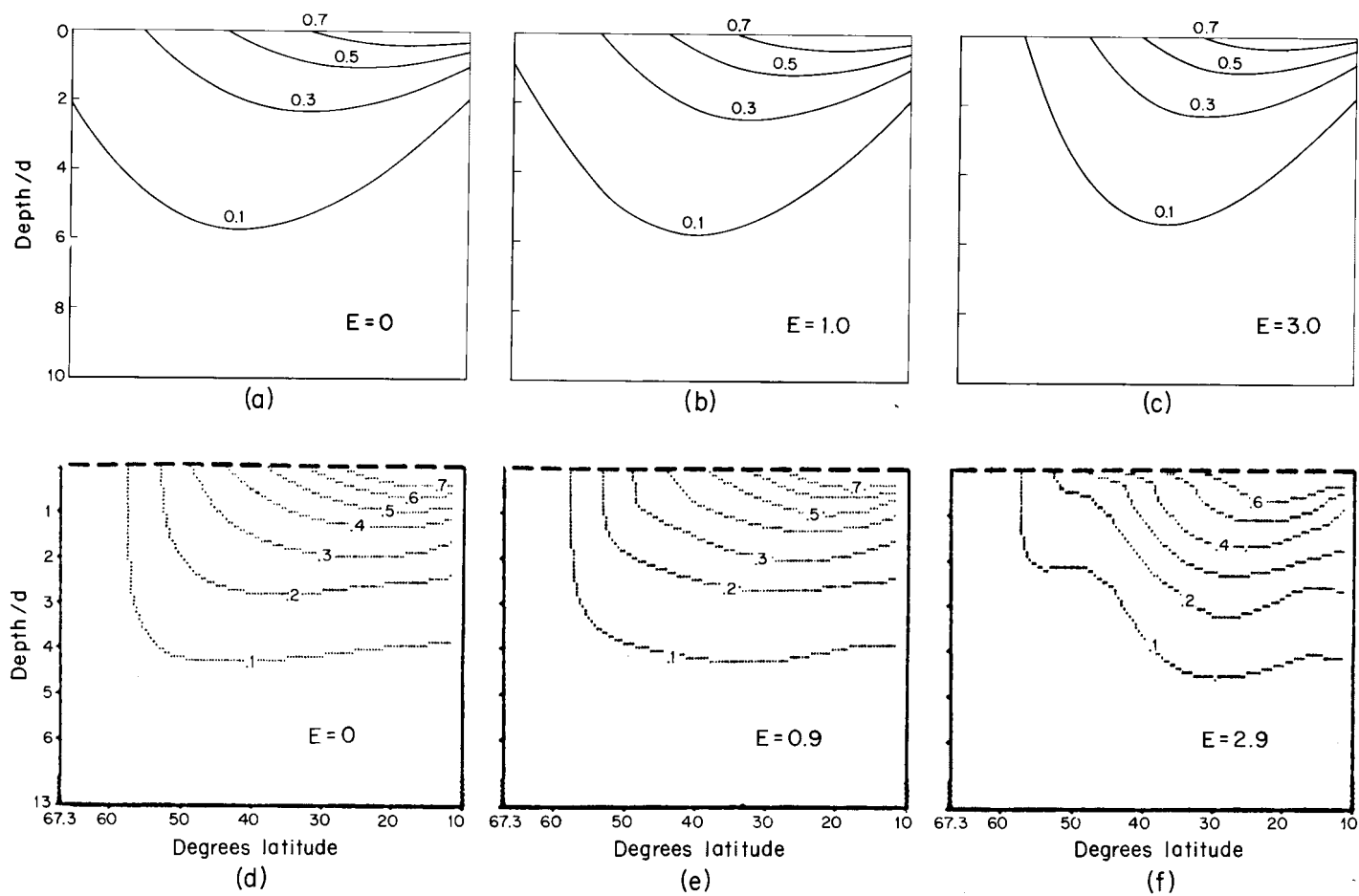
Fig. 4 -- North/south temperature sections bisecting the basin, (a through c) analytical model, (d through f) Bryan and Cox [1967]. From left to right E = 0 corresponds to no wind, E = 1 to moderate wind, and E = 3 to strong wind. The parameters K = 1 and C = 0.2 are held fixed.

3. In the case of pure thermohaline circulation (E = 0) isotherms are more nearly parallel to latitude circles in both models (Fig. 2a and 2d). However, there is a tendency for isotherms to slope slightly equatorward to the east in the analytical model which is not noted in the numerical model.

4. Quantitative discrepancies in the values of temperature are noted. This is due to the normalizing problem discussed in Section III.

Other tests of the analytical model were made. For example, the values of terms in the heat-balance equation, Eq. (2.6), were compared with those given by Bryan and Cox [1968]. Good qualitative agreement was found in the subtropics. Discussion of this is postponed to a later section to help explain other results. (See Fig. 6.)

VARIATION OF THE THERMAL DIFFUSION PARAMETER, K

Except in high-latitude regions of convective instability the value K = 1 was held fixed throughout the numerical studies reported by Bryan and Cox [1967, 1968]. The results of their 1968 study suggest that diffusion is unimportant below the surface layer in middle latitudes. It is a relatively simple matter to investigate and interpret the effects of K in the analytical model.

Holding E = 1 and C = 0.2 fixed, we set K = 0, 1, 3. In doing so we regard the depth scale D as fixed and the eddy diffusivity κ as variable. (The scale d depends on κ.) One way to do this is to define $D = (\omega a^2 / \rho_0 \alpha \Delta \theta g)^{1/2}$. In this way D is independent of κ, and E = 1 by definition. The resulting surface temperature distributions are shown in Fig. 5, in which Fig. 2b (K = 1) is repeated in order to facilitate comparisons.

The most striking feature is the lack of any significant difference in results for the cases K = 0 and K = 1. With nearly identical surface temperature distributions and with C fixed it follows that all vertical sections will look nearly the same for the two cases.

More detailed comparisons of the three cases show that the general effect of increasing K is to make surface water colder in middle latitudes. This is physically reasonable because surface water is being cooled at an increased rate by vertical conduction.
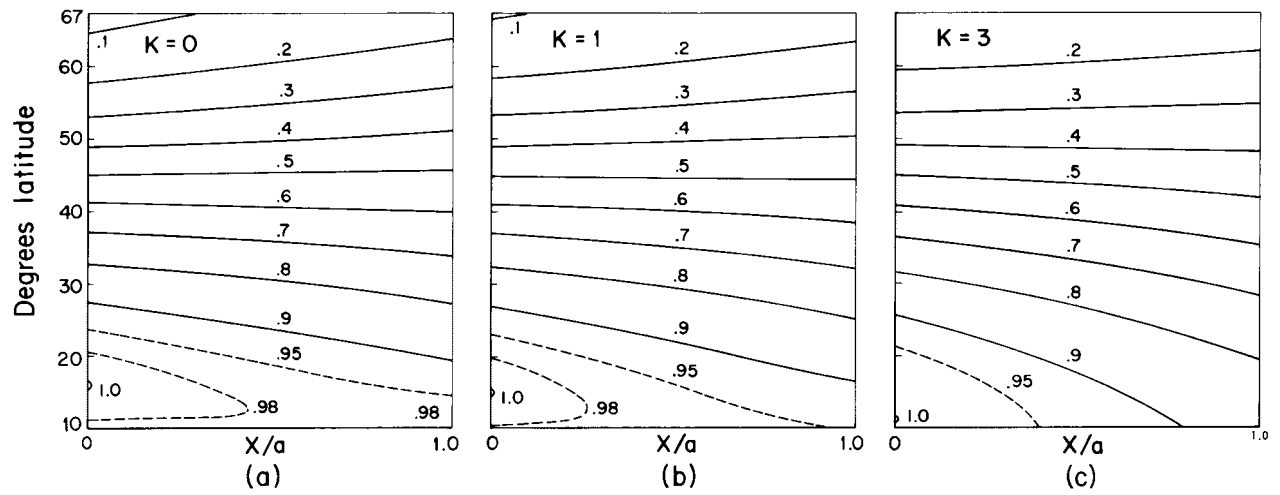
Fig. 5 -- Horizontal temperature patterns at z = 0. From left to right, K = 0 corresponds to no diffusion, K = 1 to moderate diffusion, and K = 3 to strong diffusion. The parameters E = 1 and C = 0.2 are held fixed. Note the lack of appreciable difference among the distributions.

A warming trend for increasing K can be noted to the northwest. However, its applicability to the real ocean is doubtful because the model may break down in high latitudes.

The present results showing little difference between the cases K = 0 and K = 1, together with the numerical results of Bryan and Cox [1968], offer evidence of the validity of the purely advective model of the thermocline as originally formulated by Welander [1959], provided the model is interpreted as a first approximation applying to the interior circulation in middle latitudes in intermediate waters. Higher order effects giving rise to deep upwelling are crucial for studies of the abyssal circulation.

HEAT-BALANCE COMPONENTS

Let an overbar denote the east/west average of a quantity. Then

$$K \overline{\frac{\partial^2 \theta}{\partial z^2}} = \frac{KC^2}{f^2} \overline{\theta}_o e^{Cz/f} \quad , \tag{4.1}$$

$$\overline{-\underset{\sim}{v}_H \cdot \nabla_H \theta} = \frac{\beta}{2Cf} \overline{\frac{\partial \theta_o}{\partial x}}^2 e^{2Cz/f} \quad , \tag{4.2}$$

$$-\overline{w\frac{\partial \theta}{\partial z}} = \overline{\underset{\sim}{v}_H \cdot \nabla_H \theta} - K\overline{\frac{\partial^2 \theta}{\partial z^2}} \quad . \tag{4.3}$$

The above for K = E = 1 and C = 0.2 are shown in Fig. 6 plotted versus depth at 33°N. For comparison, Bryan and Cox's [1968] corresponding results are shown. Qualitative agreement is good below the maximum of $\overline{-w\partial\theta/\partial z}$ in the numerical result. Just below this point, corresponding roughly to the surface in the analytical model, the advection terms are dominant and diffusion is relatively small. At greater depths all terms are much smaller and the dominant balance is between vertical advection and vertical diffusion.
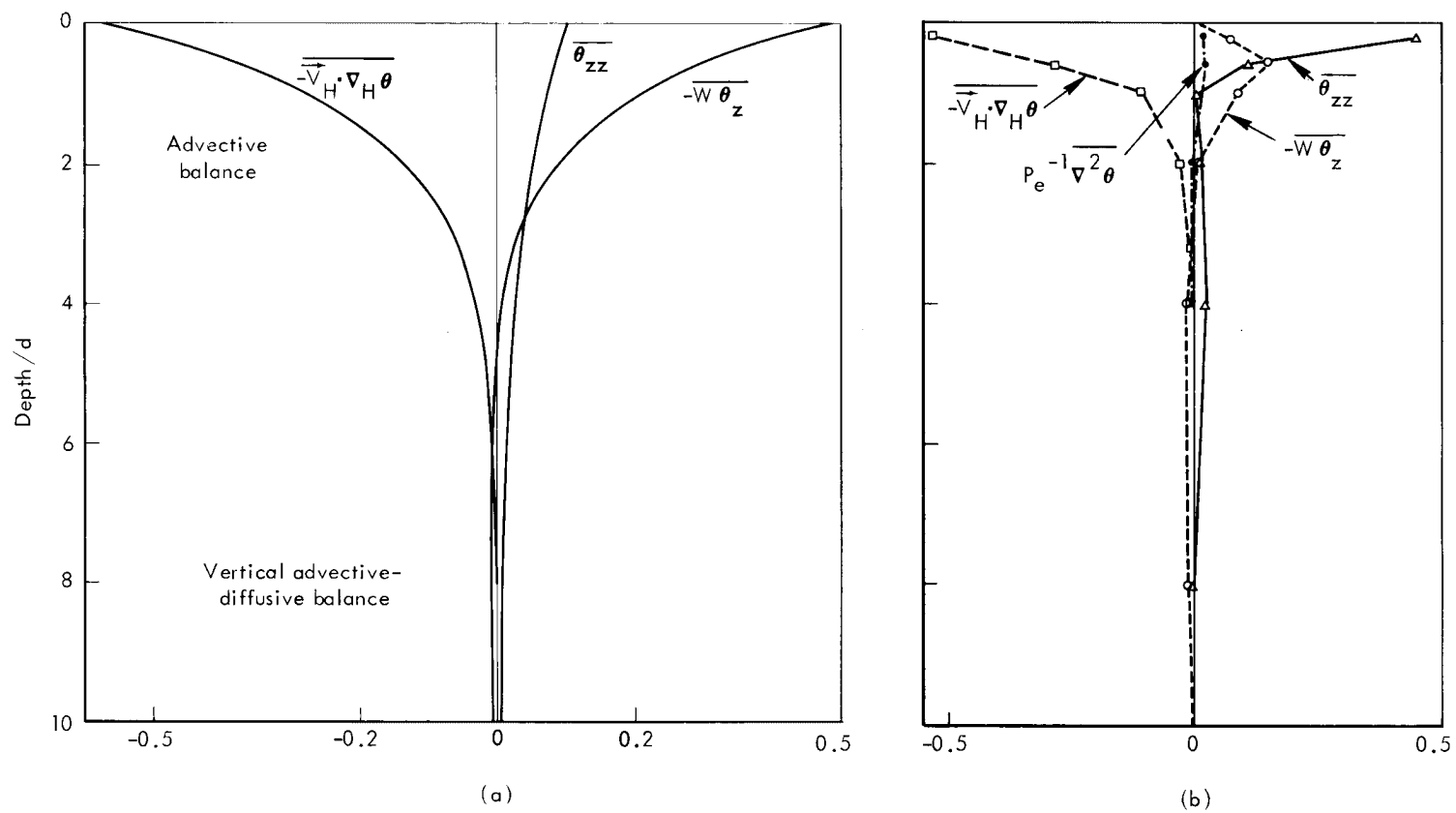
Fig. 6 -- East/west average of heat balance components at 33°N for K = E = 1, C = 0.2; (a) analytical model, (b) Bryan and Cox [1968]. The surface z = 0 in the analytical model corresponds roughly to the point where $-\overline{w\theta_z}$ is a maximum in the numerical model.

Further insight is gained by taking the ratio of Eq. (4.2) to
Eq. (4.1). Writing the result in terms of the stress function and the
parameters, and taking into account the negative sign of the stress
function (~Ekman-layer convergence) in middle latitudes, one finds

$$\frac{\overline{\underline{v}_H \cdot \nabla_H \theta}}{K \overline{\theta_{zz}}} = [1 + \frac{E}{KC} \cot \phi \, |F(\phi)|] \, e^{Cz/f} \quad . \qquad (4.4)$$

The product of the cotangent and the stress function is of order
unity in middle latitudes. Near the surface $z = 0$ the above ratio is
large because

$$\frac{E}{KC} = \frac{E_c}{K_c} >> 1 \quad , \qquad (4.5)$$

where E_c and K_c denote parameters based on the exponential-scale depth
rather than D. [Note that $E \sim 1/D^2$ and $K \sim 1/D^3$, that the products
EC^2 and KC^3 appear in Eq. (2.18), and that the ratio, Eq. (4.5), can
be written EC^2/KC^3.] In the reference case this ratio is 5. An in-
tuitive explanation is that the thermal boundary layer is not suffi-
ciently thin to make vertical conduction important for a fixed vertical
temperature difference.

A second result from Eq. (4.4) is that, for all values of the pa-
rameters, the ratio of horizontal advection to vertical diffusion is
never less than one at the surface in regions where the stress function
is negative or zero, i.e. in middle latitudes. In the thermohaline
limit $E = 0$ the ratio is one because the Ekman velocity, and hence also
vertical advection, is zero at the surface. In this limit, vertical
diffusion when present must be balanced by horizontal advection (pro-
vided neglected terms in the heat-balance equation are truly negligible).

For all values of the parameters except $K = 0$ the ratio, Eq. (4.4),
tends exponentially to zero with depth. Consistent with earlier results
[Robinson and Welander, 1963; Blandford, 1965], vertical diffusion when
present must be balanced at great depths by vertical advection (with

the same provision as above; see Veronis [1969] who shows that horizontal diffusion also can give rise to deep vertical advection). Horizontal advection decays exponentially at twice the rate of vertical diffusion [c.f. Eqs. (4.1) and (4.2)] because the former is the product of an exponentially decaying velocity and an exponentially decaying temperature gradient.

Heat-balance components for the purely thermal case $E = 0$, $K = 1$, $C = 0.2$ were computed but are not shown. This case is found qualitatively by displacing the surface in Fig. 6a downward to a depth of approximately 4.7 where vertical advection passes through zero. All components are much smaller for this case. While horizontal advection is relatively important at the surface, it decays rapidly and the deep vertical-advection/vertical-diffusion balance shown in Fig. 6 is achieved at shallower depths. This may explain why Robinson and Welander [1963], who considered only the case of thermal circulation, found that horizontal advection played a relatively minor role, whereas Blandford [1965], whose solution contained nonvanishing vertical velocities at the surface and therefore corresponded more nearly to the present case with $E = 1$, found that horizontal advection played a relatively important role.

V. CONCLUSIONS AND FURTHER DISCUSSION

Tests of the calibrated analytical model in which the wind-stress
parameter E is varied give temperature distributions that are in qual-
itative agreement with those of the numerical model of Bryan and Cox
[1967] for the region beneath the surface layer in middle latitudes of
the open ocean. Probably the most striking feature that occurs in both
models for increasing E is the increasingly pronounced subtropical high-
temperature pattern (cf. Fig. 2).

Heat-balance components in the subtropics for the reference case
E = K = 1, C = 0.2, are in qualitative agreement with Bryan and Cox
[1968] below the depth where downward advection is a maximum in
the numerical model (cf. Fig. 6). For this region, both models indi-
cate that the dominant balance is between horizontal and vertical ad-
vection. Vertical diffusion plays a relatively minor role in the ref-
erence case except at great depths. As noted above, discrepancies
between earlier thermocline models [Robinson and Welander, 1963; Bland-
ford, 1965] are probably related to the presence or absence of wind-
induced downwelling at the surface.

For more general choices of the parameters E, K, and C, the fore-
going analytical model indicates that horizontal advection of (apparent)
heat is probably always at least as important as vertical diffusion in
the thermocline region in middle latitudes.

It is known that certain results are not sensitive to values of
the vertical thermal-diffusion coefficient in the order of magnitude
range $0.1 < \kappa < 10$ cm^2/sec [e.g., Bryan and Cox, 1967; Veronis, 1969].
For example, note from Table 1 that the thermal-diffusion scale $d \propto \kappa^{1/3}$
is a slowly varying function of κ. This lack of sensitivity is borne
out by the foregoing analysis in terms of temperature distributions
(cf. Fig 5). Perhaps one of the more useful new results is the lack
of appreciable difference in temperature distributions even when K = 0
(E and C fixed). The above results, together with those of Bryan and
Cox [1968], indicate that Welander's [1959] purely advective model,
suitably interpreted, is a correct first approximation for the inter-
mediate circulation.

The validity of the purely advective model is further borne out by the near-constancy of potential vorticity along lines of constant θ in middle latitudes (which is strictly true only for $K = 0$). Table 2 indicates that $(f/\theta)\partial\theta/\partial z$ is more nearly a weak function of θ (for latitudes $\phi > 10°N$) than an absolute constant as assumed here. Future improvements in analytical models might be made by calibrating more general solutions to the potential vorticity equation with respect to oceanographic data.

Quantitative discrepancies in the values of temperature and heat-balance components from the analytical and numerical models are probably not serious (cf. the normalizing problem discussed in Section III). These discrepancies are partly related to surface-temperature boundary conditions imposed on the "first generation" numerical models. It appears likely that closer agreement will be found between "second generation" analytical and numerical models as more realistic surface conditions are assumed. [See, e.g., Bryan, 1969.]

Perhaps a more serious difficulty in the foregoing analytical model is the neglect of the barotropic mode (pressure and velocity components independent of depth). It appears that the barotropic mode is related to the finite depth of the ocean and to associated topographic effects [Fofonoff, 1962]. The results of the foregoing analysis suggest that barotropic effects may be small; however, both the calibrated model and the numerical models on which it is based neglect bottom topography. It is not clear, for example, how the barotropic advection of heat in a more realistic ocean model might change the heat balance depicted in Fig. 6. As results from more realistic numerical models become available, future work on analytical models might be directed at including the barotropic mode. The task will be more difficult because the Ekman velocity condition becomes nonlinear in the derivatives of baroclinic and barotropic components of the unknown pressure field, but it is not hopeless [cf. Eq. (8.6) of Veronis, 1969].

As our confidence in the analytical models increases, it is possible that a suitably calibrated model can be used to parameterize the interior circulation in order to facilitate more detailed investigations of certain of the boundary regimes. For example, in numerical investigations of

the oceanic general circulation, a denser network of grid points in the surface layer would likely become feasible if the circulation beneath that layer were prescribed analytically over a considerable part of the ocean.

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